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THE INVARIANT SUBSPACE PROBLEM FOR CONTRACTIVE OPERATORS IN KREIN SPACES

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Abstract. The invariant subspace problem, which is one of the most fundamental questions in operator theory and which has been a subject of study for several decades remains open even on a Hilbert space. However, for certain classes of operators, for example, compact operators, solutions do exist. In this paper, we address this question for contractive operators T defined on a Krein space $\mathcal{K}$. We investigate the existence of semi-definite invariant subspaces and establish that every contractive operator T defined on a Krein space $\mathcal{K}$ has maximal semi-definite invariant subspaces.

1. Introduction

Let A be an everywhere defined bounded linear operator acting on a normed linear space E. We say that a subspace F of E is invariant under A if $AF \subseteq F$. The invariant subspace problem, which is one of the most fundamental questions in operator theory and which has been a subject of study for several decades is the following: Does there exist a closed invariant subspace F of E under A besides the trivial ones $F = E$ and $F = \{0\}$? Let us note here that this problem is still unsolved on a Hilbert space in general. However, for certain classes of operators, for example, compact operators, solutions do exist.

Operators without invariant subspaces were first found independently by P. Enflo [6] and C. J. Read [11] on an unknown Banach space. Counter-example on the spaces ℓ_1 and c_0 were later established by Read, [12], [13]. A general account of the theory of invariant subspaces written before these counter-examples were discovered can be found in [10].

In this Paper, we address the question of invariant subspace for contraction operators T defined on a Krein space $\mathcal{K}$.

2. Indefinite inner Product Spaces

2.1. **Introduction.** The theory of indefinite inner product spaces goes back to the 1942 paper of Dirac on quantum field theory [4]. By definition an inner product space is a complex linear space together with a Hermitian form defined on it so that the associated quadratic form takes both positive and negative

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values. A very important example of an indefinite inner product space arises when one considers an orthogonal direct sum of two Hilbert spaces, one equipped with the original inner product and the other one with -1 times the original inner product.

A first mathematical treatment of indefinite inner product spaces appeared in Pontryagin's paper of 1944 [9]. Pontryagin, who was inspired by the un published work of Sobolev was unaware of similar investigation that was being carried out by Dirac at around the same time.

2.2. **Indefinite inner Product Spaces.** Let $\mathcal{H}$ be a complex linear space. By an indefinite inner product on $\mathcal{H}$ we mean a complex valued function $\langle .,. \rangle_{\mathcal{H}}$ defined on $\mathcal{H} \times \mathcal{H}$ which satisfies the following axioms:

(i) (Linearity): $\langle \alpha x + \beta y, z \rangle_{\mathcal{H}} = \alpha \langle x, z \rangle_{\mathcal{H}} + \beta \langle y, z \rangle_{\mathcal{H}}$ for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,
(ii) (Symmetry): $\langle x, y \rangle_{\mathcal{H}} = \overline{\langle y, x \rangle}_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$.

The relations in (i) and (ii) imply that

$$\langle z, \alpha x + \beta y \rangle_{\mathcal{K}} = \overline{\alpha} \langle z, x \rangle_{\mathcal{H}} + \overline{\beta} \langle z, y \rangle_{\mathcal{H}}$$

A linear space $\mathcal{H}$ together with an indefinite inner product $\langle .,. \rangle_{\mathcal{H}}$ defined on it will be called an indefinite inner product space. We shall denote this by $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$, or just $\mathcal{H}$ if is clear from the context what we mean.
Let ($\mathcal{H}, \langle .,. \rangle_{\mathcal{H}}$) be an indefinite inner product space. By the anti space of $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ we mean the space $(\mathcal{H}, -\langle .,. \rangle_{\mathcal{H}})$ which coincides with $\mathcal{H}$ as a vector space but with the sign of the indefinite inner product reversed.
A linear space $\mathcal{H}$ is said to be a direct sum of its subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, written as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ if each $h \in \mathcal{H}$ has a unique representation $h = h_1 + h_2$ with $h_1 \in \mathcal{H}_1$, and $h_2 \in \mathcal{H}_2$. Two vectors f and g in an indefinite inner product $\mathcal{H}$ are said to be orthogonal if $\langle f, g \rangle_{\mathcal{H}} = 0$. If $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ is an indefinite inner product space, its subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be orthogonal if $\langle h_1, h_2 \rangle_{\mathcal{H}} = 0$ for each $h_1 \in \mathcal{H}_1$, and $h_2 \in \mathcal{H}_2$.
An indefinite inner product space $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ is said to be non degenerate if the only vector f in $\mathcal{H}$ is such that $\langle f, g \rangle_{\mathcal{H}} = 0$ for all vectors $g \in \mathcal{H}$ is f=0.
Let $\mathcal{H}$ be an indefinite product space and let $\mathcal{M}$ be its subspace. We say that $\mathcal{M}$ is:

(i) *Positive* if $\langle f, f \rangle_{\mathcal{H}} > 0$ for all $f \in \mathcal{M}$,
(ii) *Negative* if $\langle f, f \rangle_{\mathcal{H}} < 0$ for all $f \in \mathcal{M}$,
(iii) *Definite* if it is either positive or negative,
(iv) *Indefinite* if it is neither positive nor negative,
(v) *Neutral* if $\langle f, f \rangle_{\mathcal{H}} = 0$ all $f \in \mathcal{M}$.

2.3. **Krein spaces.** By definition a Krein space is an indefinite inner product space $(\mathcal{K}, \langle .,. \rangle_{\mathcal{K}})$ which can be expressed as a direct orthogonal sum

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-, \tag{2.1}$$

where $\mathcal{K}_+$ is a Hilbert space and $\mathcal{K}_-$ is the anti space of a Hilbert space. Any such representation is called a *fundamental decomposition*. This decomposition is not unique in general. However, the components $\mathcal{K}_+$ and $\mathcal{K}_-$ in (2.1) are uniquely determined.
For a Krein space $\mathcal{K}$, the numbers $\mathrm{ind}_{\pm}\mathcal{K} = \dim \mathcal{K}_{\pm}$ do not depend on the choice of the decomposition (2.1) above. We call $\mathrm{ind}_{\pm}\mathcal{K}$ the positive and negative indices of the Krein space $\mathcal{K}$. A Krein space $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ in which $\dim \mathcal{K}_- < \infty$ is called a Pontryagin space.
Decomposition (2.1) induces a topology on the Krein $\mathcal{K}$ space in the following way: First, one forms the *associated Hilbert space* $\mid \mathcal{K} \mid := \mathcal{K} \oplus \mid \mathcal{K}_- \mid$ by replacing $\mathcal{K}_-$ with its anti space $\mid \mathcal{K}_- \mid$ which is a Hilbert space. The Hilbert space norm

$\| \, . \, \|_{|\mathcal{K}|}$ is called the norm for the Krein space $\mathcal{K}$ and all notions of convergency and continuity are understood to be with respect to this norm topology.
The operator $J_{\mathcal{K}} f = f_+ - f_-$ where $f = f_+ + f_-$ with $f_\pm \in \mathcal{K}\pm$ is called the fundamental symmetry or signature operator associated with decomposition (2.1). If $| \, \mathcal{K} \, |$ is a Hilbert space associated with decomposition (2.1) and $[.,.]_{|\mathcal{K}|}$ denotes the corresponding positive definite inner product then

$$[f \, , \, g]_{|\mathcal{K}|} = \langle J_{\mathcal{K}} f, g \rangle_{\mathcal{K}}$$

for any vectors $f, g \in \mathcal{K}$. This follows from

$$\begin{aligned} \langle J_{\mathcal{K}} f, g \rangle &= \langle f_+ - f_- \, , g_+ + g_- \rangle \\ &= \langle f_+, g_+ \rangle - \langle f_-, g_- \rangle \\ &= [f, g]_{\mathcal{K}}. \end{aligned}$$

The following are examples of Krein spaces.

Example 2.1. Consider the space $\mathbb{C}^3$ with an indefinite inner product $\langle \mathbf{x} \, , \, \mathbf{y} \rangle = x_1\overline{y}_1 + x_2\overline{y}_2 - x_3\overline{y}_3$ defined on it where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. Then $(\mathbb{C}^3, \langle \cdot, \cdot \rangle)$ is a Krein with one possible decomposition given by

$$\mathbb{C}^3 = \mathbb{C}^3_+ \oplus \mathbb{C}^3_-,$$

where $\mathbb{C}^3_+$ consists of those elements of $\mathbb{C}^3$ of the form $(x_1, x_2, 0)$ while $\mathbb{C}^3_-$ consists of those elements of $\mathbb{C}^3$ of the form $(0, 0, x_3)$.

Example 2.2. As another example, we consider the space ℓ_2, which is a vector space of all sequences $\{\xi_i\}_{i=1}^{\infty}$ of complex numbers satisfying

$$\sum_{i=1}^{\infty} |\xi_i|^2 < \infty.$$

The space

$$(\ell_2, \langle \cdot, \cdot \rangle)$$

with

$$\langle \xi, \eta \rangle = \sum_{i=1}^{\infty} (-1)^i \xi_i \overline{\eta}_i$$

where $\xi = \{\xi_i\}_{i=1}^{\infty}$ and $\eta = \{\eta_i\}_{i=1}^{\infty}$ are in ℓ_2 is a Krein space with one possible decomposition given by

$$\ell_2 \, = \, \ell_{2+} \, \oplus \, \ell_{2-},$$

where ℓ_{2+} is the space consisting of elements in ℓ_2 of the form

$$\xi_+ = \{0, \xi_2, 0, \xi_4, 0, \xi_6, ...\}$$

and ℓ_{2-} is the space consisting of those elements of ℓ_2 of the form

$$\xi_- = \{\xi_1, 0, \xi_3, 0, \xi_5, ...\} \, .$$

For a more detailed theory on Krein spaces, we refer to [1], [2], [3], [5] and [8].

2.4. **Bounded linear operators.** The notions of a bounded linear operator on any given Krein space are similar to those on a Hilbert space and in general to those of a bounded linear operator on any normed linear space.
Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces. By $B(\mathcal{H},\mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H}=\mathcal{K}$, we write $B(\mathcal{H})$ in place of $B(\mathcal{H},\mathcal{H})$. For every $A\in B(\mathcal{H},\mathcal{K})$, there is a unique $A^*\in B(\mathcal{K},\mathcal{H})$ such that $\langle Af,g\rangle_{\mathcal{K}}=\langle f,A^*g\rangle_{\mathcal{H}}$, $f\in\mathcal{H}, g\in\mathcal{K}$. We call A^* an adjoint operator of A. An operator $A\in B(\mathcal{H})$ is self adjoint if $A^*=A$. A linear operator A in an inner product space $\mathcal{K}$ is said to be isometric if $\langle Ax\,,\,Ay\rangle=\langle x,\ y\rangle$, for every pair $x\,,y\in D(A)$.

3. The Riesz projector and spectral decomposition

3.1. **Operators and operator valued functions.** If

$$P(\lambda)=\sum_{i=1}^{n}\alpha_i\lambda^i=\alpha_1\lambda+\alpha_2\lambda^2+\ldots+\alpha_n\lambda^n$$

is a polynomial with complex coefficients and T is a bounded linear operator on a complex linear space $\mathcal{H}$, then by $P(T)$ we mean the sum

$$P(T)=\sum_{i=1}^{n}\alpha_iT^i=\alpha_1T+\alpha_2T^2+\ldots+\alpha_nT^n.$$

The study of polynomials in an operator T on a finite dimensional unitary space leads to a rather complete description of the analytic behavior of the operator, and at the same time furnishes a clear geometric picture of the manner in which the operator transforms the unitary space on which it acts. In attempting to make a corresponding study of an operator T on an infinite dimensional linear space, one is immediately confronted with the necessity of introducing an algebra larger than that consisting of the polynomials in T. The development of the spectral theory in the finite dimensional space suggests that a useful definition of a function $f(T)$ of an operator T is given by the Cauchy formula

$$f(T)=\frac{1}{2\pi i}\int_C f(\lambda)(T-\lambda I)^{-1}d\lambda$$

in which f is an analytic scalar valued function and C is a suitable contour. In giving a meaning to this formula, one is naturally led to the study of questions concerning the existence and properties of the function

$$R(\lambda)=(T-\lambda I)^{-1}. \tag{3.1}$$

In order to understand (3.1) better, we recall some basic facts from the spectral theory of a bounded linear operator. Let $T:\mathcal{H}\to\mathcal{H}$ be a bounded linear operator acting on a complete normed linear space $\mathcal{H}$. By definition the *resolvent set* $\rho(T)$ of T is a set of all complex numbers λ such that for each $y\in\mathcal{H}$ the equation $Tx-\lambda x=y$ has a unique solution $x\in\mathcal{H}$. Equivalently, $\lambda\in\rho(T)$ if and only if $T-\lambda I$ is an invertible operator, that is, there is a bounded linear operator $R(\lambda)$ on $\mathcal{H}$ such that

$$R(\lambda)(T-\lambda I)=(T-\lambda I)R(\lambda)=I \tag{3.2}$$

The compliment of $\rho(T)$ in $\mathbb{C}$ is called the *spectrum* of T and is denoted by $\sigma(T)$. It is well known that $\sigma(T)$ is a bounded closed subset of $\mathbb{C}$. The operator $R(\lambda)$ appearing in (3.2) will be denoted by $(T-\lambda I)^{-1}$ and is called the *resolvent* of T. In what follows some basic theorems of complex analysis are extended to vector and operator valued functions. We start with the definition of contour integrals of the form

$$\frac{1}{2\pi i}\int_{\Gamma} g(\lambda)d\lambda \tag{3.3}$$

where the integrand is a function with values in some complete normed linear space. First let us make clear what kind of contours are used in (3.3). We call Γ a *Cauchy contour* if Γ is the oriented boundary of a bounded Cauchy domain in $\mathbb{C}$. By definition, a *Cauchy domain* is a disjoint union in $\mathbb{C}$ of a finite number of non-empty open connected sets $\Delta_1, ..., \Delta_r$, say, such that $\overline{\Delta}_i \bigcap \overline{\Delta}_j = \theta (i \neq j)$ and for j the boundary of Δ_j consists of a non intersecting closed rectifiable Jordan curves which are oriented in a such away that Δ_j belongs to the inner domains of the curves. If σ is a compact subset of a (nonempty) open set $\Omega \subset \mathbb{C}$, then we can always find a Cauchy contour Γ in Ω such that σ belongs to the inner domain of Γ.

Let Γ be a Cauchy contour, and let $g : \Gamma \to \mathcal{H}$ be a continuous function on Γ with values in a complete normed linear space $\mathcal{H}$. Then (as in complex function theory) the integral (3.3) is defined as a Stieltjes integral, but now its convergence has to be understood in the norm of $\mathcal{H}$. Thus the value of (3.3) is a vector in $\mathcal{H}$ which appears as a limit (in the norm of $\mathcal{H}$) of the corresponding Stieltjes sum. From this definition, it clear that

$$F\left(\frac{1}{2\pi i}\int_{\Gamma} g(\lambda)d\lambda\right) = \frac{1}{2\pi i}\int_{\Gamma} F(g(\lambda))d\lambda \tag{3.4}$$

for any continuous linear functional F on $\mathcal{H}$. Note that the integrand of the second integral in (3.4) is just a scalar valued function. Often the integral in (3.3) can be computed if g has additional properties.

We now consider the integral (3.3) for the case when $\mathcal{H}$ is the Banach space $\mathcal{L}(X, Y)$ consisting of all bounded linear operators from the Banach space X into the Banach space Y. Let X and Y be Banach spaces and let $g : \Gamma \to \mathcal{L}(X, Y)$ be a continuous function. Then the value of the integral (3.3) is abounded linear operator from X into Y and for each $x \in X$ we have

$$\left(\frac{1}{2\pi i}\int_{\Gamma} g(\lambda)d\lambda\right) x = \frac{1}{2\pi i}\int_{\Gamma} g(\lambda)x d\lambda. \tag{3.5}$$

Further more, if $A : X_1 \to X$ and $B : Y \to Y_1$ are bounded linear operators acting between Banach spaces, then

$$B\left(\frac{1}{2\pi i}\int_{\Gamma} g(\lambda)d\lambda\right) A = \frac{1}{2\pi i}\int_{\Gamma} Bg(\lambda)A d\lambda \tag{3.6}$$

We note that all the work in this section can be found in [7]

3.2. **Spectral decomposition and the Riesz projection.** Let T be a bounded linear operator on a Banach space X. If $\mathcal{M}$ is a subspace of X invariant under T, then $T|_{\mathcal{M}}$ denotes the *restriction* of T to $\mathcal{M}$, which has to be considered as an operator from $\mathcal{M}$ to $\mathcal{M}$.
Assume that the spectrum of T is a disjoint union of two non-empty closed subsets Ω and Φ. We want to show that to this decomposition of the spectrum there corresponds a direct sum decomposition $X = M \oplus N$ such that M and N are T-invariant subspaces of X, the spectrum of the restriction $T|_{\mathcal{M}}$ is precisely is equal to Ω and that of $T|_{\mathcal{N}}$ to Φ. To prove that such a spectral decomposition exists we study the operator

$$P = -\frac{1}{2\pi i}\int_{\Gamma}(T-\lambda I)^{-1}d\lambda. \tag{3.7}$$

A set Ω is called an *isolated part* of the spectrum $\sigma(T)$ of an operator T if both Ω and $\Phi := \sigma(T)\backslash\Omega$ are closed subsets of $\sigma(T)$. Given an isolated part Ω of $\sigma(T)$ we define P_Ω to be a bounded linear operator on X given by the right hand of (3.7), where we assume that Γ is a Cauchy contour (in the resolvent set of T) around Ω separating Ω from $\Phi = \sigma(T)\backslash\Omega$. By the later we mean that Ω belongs to inner domain of Γ and Φ to the outer domain of Γ. Since $(T-\lambda I)^{-1}$ is an analytic operator function (in λ) on the resolvent set of T, a standard argument of complex function theory shows that the definition of P_Ω does not depend on the particular choice of the contour Γ.
The operator P_Ω defined above is called the *Riesz projection* of T corresponding to the isolated part Ω. The use of the word projection is justified by Lemma 3.2 below.

Lemma 3.1. *The following identity, known as the resolvent identity, is valid for every pair $\lambda,\ \mu\ \in \rho(T)$, the resolvent set of T:*

$$(T-\lambda I)^{-1} - (T-\mu I)^{-1} = (\lambda-\mu)(T-\lambda I)^{-1}(T-\mu I)^{-1}.$$

Proof. The lemma follows by multiplying both sides of the equation

$$\begin{aligned}(T-\mu I)(T-\lambda I)[(T-\lambda I)^{-1} - (T-\mu I)^{-1}] &= (T-\mu I)-(T-\lambda I)\\ &= (\lambda-\mu)I\end{aligned}$$

by $(T-\lambda I)^{-1}(T-\mu I)^{-1}$ □

Lemma 3.2. *The operator*

$$P_\Omega = -\frac{1}{2\pi i}\int_{\Gamma}(T-\lambda I)^{-1}d\lambda$$

as defined above is a projection, that is, $P_\Omega^2 = P_\Omega$.

Proof. Let Γ_1 and Γ_2 be Cauchy contours around Ω separating Ω from $\Phi = \sigma(T)\backslash\Omega$ and assume that Γ_1 is in the inner domain of Γ_2. Then

$$\begin{aligned}P_\Omega^2 &= \left(\frac{1}{2\pi i}\int_{\Gamma_1}(T-\lambda I)^{-1}d\lambda\right)\left(\frac{1}{2\pi i}\int_{\Gamma_2}(T-\mu I)^{-1}d\mu\right)\\ &= \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_1}\int_{\Gamma_2}(T-\lambda I)^{-1}(T-\mu I)^{-1}d\mu d\lambda.\end{aligned}$$

We now use resolvent identity in Lemma 3.1 and write

$$P_\Omega^2 = Q - R,$$

where

$$\begin{aligned}
Q &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{1}{\lambda-\mu}(T-\lambda I)^{-1} d\mu d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} (T-\lambda I)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{1}{\lambda-\mu} I d\mu\right) d\lambda \\
&= -\frac{1}{2\pi i}\int_{\Gamma_1}(T-\lambda I)^{-1} d\lambda \\
&= P_\Omega.
\end{aligned}$$

and

$$\begin{aligned}
R &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{1}{\lambda-\mu}(T-\mu I)^{-1} d\mu d\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2}\int_{\Gamma_1} \frac{1}{\lambda-\mu}(T-\mu I)^{-1} d\lambda d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} (T-\lambda I)^{-1}\left(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{\lambda-\mu} I d\lambda\right) d\mu \\
&= 0.
\end{aligned}$$

We have used the fact that

$$\int_{\Gamma_2}\frac{d\lambda}{\mu-\lambda} = -2\pi i,\ \lambda\in\Gamma_1,$$

$$\int_{\Gamma_1}\frac{d\lambda}{\mu-\lambda} = 0,\ \mu\in\Gamma_2$$

and these identities hold true because Γ_1 is in the inner domain of Γ_2. Further more, the interchange of the integrals in the computation for R is justified by the fact that the integrand is a continuous operator function on $\Gamma_1 \times \Gamma_2$. □

Theorem 3.3. *Let T be a bounded linear operator acting on a Banach space X and let Ω be an isolated part of $\sigma(T)$. Let $M = P_\Omega X$ and let $N = (I - P_\Omega)X$. Then*

$$X = M \oplus N$$

and the spaces M and N are T-invariant subspaces of X.

Proof. Since P_Ω is a projection, it is clear that M and N are closed subspaces of X and

$$X = M \oplus N.$$

For any $\lambda \in \rho(T)$, the equality

$$T(T-\lambda I)^{-1} = (T-\lambda I)^{-1}T \tag{3.8}$$

holds, that is, T commutes with its resolvent operator. We multiply each side of (3.8) and integrate around a suitable contour to get

$$TP_\Omega = P_\Omega T,$$

which implies that M and N are invariant under T. $\square$

4. The Invariant subspace Problem for contractive operators in Krein spaces

4.1. **Introduction.** Let $\mathcal{K}$ be a Krein space with indefinite inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ defined on it. This indefinite inner product gives rise to a classification of elements of $\mathcal{K}$. An element $k \in \mathcal{K}$ is called *positive*, *negative*, or *neutral* if $\langle k, k\rangle_{\mathcal{K}} > 0$, $\langle k, k\rangle_{\mathcal{K}} < 0$, or $\langle k, k\rangle_{\mathcal{K}} = 0$ respectively. A linear manifold or subspace $\mathcal{L}$ in $\mathcal{K}$ is called *indefinite* if it contains both positive and negative vectors. We say that $\mathcal{L}$ is *semi-definite* if it is not indefinite. A semi-definite subspace $\mathcal{L}$ is called *non-negative* (*positive*, *uniformly positive*) if $[x, x] \geq 0$ ($[x, x] > 0$, $[x, x] \geq \delta\|x\|$, $(\delta > 0)$) for all x in $\mathcal{L}$. A *non-positive*, (*negative*, *uniformly negative*) subspace is defined in a similar way. We say that the subspace $\mathcal{L}$ is *definite* if $[x, x] = 0$ if and only if $x = 0$.
If a non-negative subspace $\mathcal{L}$ admits no nontrivial nonnegative extensions, then it is called a *maximal non-negative* subspace. Maximal non-positive (negative, uniformly negative) subspaces in $\mathcal{K}$ are similarly defined.
Maximal negative and Maximal positive subspaces can be characterized in terms of the angular operator.
Let $\mathcal{K}$ be a Krein space with a fixed fundamental decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-. \tag{4.1}$$

Consider a nonnegative subspace $\mathcal{M}$ of $\mathcal{K}$. Then for an arbitrary $x \in \mathcal{M}$,

$$x = x_+ + x_-, \;\; x_+ \in \mathcal{K}_+, \, x_- \in \mathcal{K}_-,$$

we have

$$\langle x, x\rangle_{\mathcal{K}} = [x_+, x_+]_{\mathcal{K}} - [x_-, x_-]_{\mathcal{K}} \geq 0,$$

that is,

$$\| x_- \|^2 \leq \| x_+ \|^2 \tag{4.2}$$

and

$$\|x_+\|^2 \leq \|x_+\|^2 + \|x_-\|^2 = \|x\|^2 \leq 2\|x_+\|^2.$$

These inequalities show that on $\mathcal{M}$, the mapping $x \mapsto x_+$ together with its inverse $x_+ \mapsto x$ are one to one and bounded. It is also clear that these mappings are also linear. We note that $x \mapsto x_+$ is bounded because

$$\|x_+\| \leq \|x\|$$

while $x_+ \mapsto x$ in bounded because

$$\| x \|^2 \leq 2 \| x_+ \|^2$$

as seen from above. To show that the mapping $T : x \mapsto x_+$ is one to one, let $x \in \ker T$, the kernel of T. Then $\| x_+ \|^2 \leq \| Tx \|^2 \leq 2 \| x_+ \|^2$. This implies that $x_+ = 0$. Inequality (4.2) then implies that $x_- = 0$ and so $x = 0$.

This shows that the mapping $T : x \mapsto x_+$ is one to one. Denoting its range by $\mathcal{M}_+ \subset \mathcal{K}_+$ and the inverse mapping by F, we have that $Fx_+ = x\,(x_+ \in \mathcal{M}_+)$ and $x_- = x - x_+ = (F - I)x_+$, if $x \in \mathcal{M}$.
Define on $\mathcal{M}_+$ the operator $K = F - I$. It follows that $x_- = Kx_+$ and $x = x_+ + Kx_+ \;\; x \in \mathcal{M}$. Inequality (4.2) implies that

$$\| Kx_+ \|^2 \leq \| x_+ \|^2$$

and that $\|K\| \leq 1$. Thus we have proved the following lemma.

Lemma 4.1. *To an arbitrary non negative subspace $\mathcal{M} \subset \mathcal{K}$ there corresponds a subspace $\mathcal{M}_+ \subset \mathcal{K}_+$, an operator K from $\mathcal{M}_+$ into $\mathcal{K}_-$, $\| K \| \leq 1$ such that the decomposition (4.1) of x is $x = x_+ + Kx_+$, $(x_+ \in \mathcal{M}_+)$*

The operator K is called the *angle operator* of the non negative subspace $\mathcal{M}$. The definition of the angular operators in Lemma (4.1) implies that its domain is $\mathcal{M}_+$.
If a non negative subspace $\widetilde{\mathcal{M}} \subset \mathcal{K}$ is an extension of $\mathcal{M}$, then evidently, the angle operator $\widetilde{K}$ of $\widetilde{\mathcal{M}}_+$ is an extension of K, that is,

$$\text{dom } \widetilde{K} = \widetilde{\mathcal{M}}_+ \supset \text{dom } K = \mathcal{M}_+$$

and $\widetilde{K}x_+ = Kx_+$ if $x_+ \in \mathcal{M}_+$. Therefore, a non negative subspace $\mathcal{M}$ of $\mathcal{K}$ has a non negative proper extension $\widetilde{\mathcal{M}}$ if and only if $\mathcal{M}_+ \neq \mathcal{K}_+$. This leads to the following theorem.

Theorem 4.2. *Let $\mathcal{M}$ be a non negative subspace of a Krein space $\mathcal{K}$. Then $\mathcal{M}$ is maximal if and only if the domain of the angular operator for $\mathcal{M}$ given in lemma (4.1) is $\mathcal{K}_+$.*

Results similar to those in this section hold for a non positive subspace $\mathcal{N}$.

4.2. **The Invariant subspace Problem for contractive operators in Krein spaces.** Here we discuss the existence of invariant maximal semi-definite subspaces for contractive operators acting on a Krein space $\mathcal{K}$ where we give a proof of theorem 4.3. First we consider some few definitions.
Let T be a bounded linear operator on a Krein space $\mathcal{K}$. We say that T is

(i) *contractive* if $\langle Tx, Tx\rangle \leq \langle x, x\rangle$ for all $x \in \mathcal{K}$,
(ii) *strictly contractive* if $\langle Tx, Tx\rangle < \langle x, x\rangle$ for all $x \in \mathcal{K}$,
(iii) *uniformly contractive* if $\langle Tx, Tx\rangle \leq \langle x, x\rangle - \sigma \| x \|^2$ for some $\sigma > 0$.

Theorem 4.3. *Let T be a contractive operator acting on a Krein space $\mathcal{K}$. If $|\xi| \neq 1$ for every $\xi \in \sigma(T)$, the spectrum of T, then $\mathcal{K}$ is a direct sum $\mathcal{K} = \mathcal{M}_+ \oplus \mathcal{M}_-$ of two subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$ invariant under T such that $\mathcal{M}_+$ is maximal non negative and $\mathcal{M}_-$ is maximal non positive.*

Proof. Let $\mid \xi \mid = 1$. Then $\xi \in \rho(T)$, the resolvent set of T and so the operator $(T - \xi I)^{-1}$ exists, where I denote the identity operator on $\mathcal{K}$. Let $y \in \mathcal{K}$ be such that $y \neq 0$. Since T is contractive we have

$$\langle T(T - \xi I)^{-1}y, T(T - \xi I)^{-1}y\rangle \leq \langle (T - \xi I)^{-1}y, (T - \xi I)^{-1}y\rangle. \tag{4.3}$$

Let $(T-\xi I)^{-1}y = z$. Then $y = (T-\xi I)z = Tz - \xi z$. Inequality (4.3) implies that

$$\langle Tz, Tz\rangle \;\leq\; \langle z, z\rangle. \tag{4.4}$$

Since $Tz = y + \xi z$, inequality (4.4) can be written as

$$\langle y+\xi z\,,\, y+\xi z\rangle \,-\, \langle z, z\rangle \leq 0.$$

We use the fact that $|\xi| = 1$ to obtain

$$\langle y, y\rangle + \xi\langle z, y\rangle + \overline{\xi}\langle y, z\rangle \leq 0,$$

that is,

$$\langle y, y\rangle + \xi\langle (T-\xi I)^{-1}y, y\rangle + \overline{\xi}\langle y, (T-\xi I)^{-1}y\rangle \leq 0. \tag{4.5}$$

We now introduce the Riesz Projector

$$P = -\frac{1}{2\pi i}\int_{|\xi|=1}(T-\xi I)^{-1}d\xi.$$

Put $\xi = e^{i\theta}$ in (4.5), multiplying both sides by $\frac{1}{2\pi}$ and integrating from 0 to 2π with respect to θ to get

$$\langle y, y\rangle + \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta}\langle (T-e^{i\theta}I)^{-1}y, y\rangle d\theta + \frac{1}{2\pi}\int_0^{2\pi}\overline{\langle e^{i\theta}(T-e^{i\theta}I)^{-1}y, y\rangle} d\theta \;\leq\; 0. \tag{4.6}$$

We can rewrite (4.6) as

$$\langle y, y\rangle + \frac{1}{2\pi i}\int_0^{2\pi} ie^{i\theta}\langle (T-e^{i\theta}I)^{-1}y, y\rangle d\theta + \overline{\frac{1}{2\pi i}\int_0^{2\pi} ie^{i\theta}\langle (T-e^{i\theta}I)^{-1}y, y\rangle d\theta} \;\leq\; 0.$$

Hence

$$\langle y, y\rangle \leq -\frac{1}{2\pi i}\int_0^{2\pi} ie^{i\theta}\langle (T-e^{i\theta}I)^{-1}y, y\rangle d\theta - \overline{\frac{1}{2\pi i}\int_0^{2\pi} ie^{i\theta}\langle (T-e^{i\theta}I)^{-1}y, y\rangle d\theta},$$

which we can rewrite as

$$\langle y, y\rangle \leq -\frac{1}{2\pi i}\int_{|\xi|=1}\langle (T-\xi I)^{-1}y, y\rangle d\xi - \overline{\frac{1}{2\pi i}\int_{|\xi|=1}\langle (T-\xi I)^{-1}y, y\rangle d\xi}.$$

We now use the fact that $\langle\,\cdot\,,\; y\;\rangle$ is a linear functional on $\mathcal{K}$, (3.4) and (3.5) to get

$$\langle y, y\rangle \leq \left\langle -\frac{1}{2\pi i}\int_{|\xi|=1}(T-\xi I)^{-1}d\xi\; y, y\right\rangle + \overline{\left\langle -\frac{1}{2\pi i}\int_{|\xi|=1}(T-\xi I)^{-1}d\xi\; y, y\right\rangle}.$$

We rewrite the above inequality as

$$\langle y, y\rangle \leq \left\langle -\frac{1}{2\pi i}\int_{|\xi|=1}(T-\xi I)^{-1}d\xi\; y, y\right\rangle + \left\langle y, -\frac{1}{2\pi i}\int_{|\xi|=1}(T-\xi I)^{-1}d\xi\; y\right\rangle$$

and obtain

$$\langle y, y\rangle \;\leq\; \langle Py, y\rangle + \langle y, Py\rangle = 2Re\langle Py, y\rangle, \tag{4.7}$$

where P is the projection operator

$$P = -\frac{1}{2\pi i}\int_{|\xi|=1}(T-\xi I)^{-1}d\xi$$

defined earlier.
The subspaces $\mathcal{M}_+ = P\mathcal{K}$ and $\mathcal{M}_- = (I-P)\mathcal{K}$ are invariant under T. Inequality (4.7) implies that for $y \in \mathcal{M}_+$, $\langle y, y\rangle \geq 0$ while $\langle y, y\rangle \leq 0$ for $y \in \mathcal{M}_-$. Hence $\mathcal{M}_+$ is a non negative invariant subspace for T while $\mathcal{M}_-$ is a non positive invariant subspace for T.
To show that $\mathcal{M}_+$ and $\mathcal{M}_-$ are maximal, we let $\mathcal{M}_+ \subset \mathcal{N}_+$ and $\mathcal{M}_- \subset \mathcal{N}_-$ where $\mathcal{N}_+$ is a non negative subspace with $\mathcal{M}_+ \neq \mathcal{N}_+$ and $\mathcal{N}_-$ is a non positive subspace with $\mathcal{M}_- \neq \mathcal{N}_-$. For $x \in \mathcal{N}_+ \backslash \mathcal{M}_+$, we have

$$x = x_+ + x_-, \quad x_+ \in \mathcal{M}_+ \subset \mathcal{N}_+,\ x_- \in \mathcal{M}_- \subset \mathcal{N}_-.$$

Hence,

$$x_- = x - x_+ \in \mathcal{N}_+.$$

On the other hand,

$$\mathcal{M}_- \cap \mathcal{N}_+ = \{0\},$$

which implies that $x_- = 0$ and $x = x_+ \in \mathcal{M}_+$, a contradiction. Hence $\mathcal{M}_+$ is maximal. A similar argument shows that $\mathcal{M}_-$ is also maximal. □

The fact that a uniformly contractive operator is contractive yields the following corollary.

Corollary 4.4. *If T is a uniformly contractive operator, then the conclusion of Theorem (4.3) still holds.*

5. Conclusion

Let T be a bi-contractive operator acting on a Krein space $\mathcal{K}$, that is both T and its adjoint T^* are contractive. If the unit circle lies in the intersection of the resolvent sets T and T^*, then from what we have shown above, both T and T^* have maximal invariant semi-definite subspaces.

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